Problem Set 7 – Relativity (`GOI36A`)

03/11/2020

A little bit of an easier problem session this week, for two reasons. The first is that last week's problem set was rather involved; if you haven't already, go look at the harder problems from last week! The second is that you have homework due next week. So, now you have no excuse not to ace it!

1 You're Killing Me, Schwarzschild

1.) Consider the Schwarzschild metric for a spherically symmetric, static spacetime in $3+1$ dimensions:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\Omega^2 \ , \tag{1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ is the usual metric on S^2. This spacetime has four linearly independent Killing vectors; write them down.

Note that there are two ways to go about this. You could brute force it by solving the full Killing vector equations. Alternatively, though, you could use what we've discussed so far about Killing vectors and isometries and see if you can write down what the Killing vectors are without actually computing them explicitly.

2.) As we proved last week, on any given geodesic $x^\mu(\lambda)$ we can associate each Killing vector with a conserved quantity of the form

$$K_\mu \frac{dx^\mu}{d\lambda} \ . \tag{1.2}$$

(If you haven't yet proved that this is conserved along the geodesic, go look at last week's problem set!) Physically, explain what the conserved quantities associated with each of the four Killing vectors in the Schwarzschild spacetime are.

2 A First Look at AdS_2

An important class of spacetimes in theoretical physics are anti-de Sitter spacetimes, referred to as AdS_d where d is the number of spacetime dimensions. These spacetimes all have Lorentzian signature and constant negative curvature. In this problem, you'll get your first look at such spacetimes by studying AdS_2 in detail.

3.) The metric on AdS_2 can be presented as

$$ds^2 = -\cosh^2\eta \, d\tau^2 + d\eta^2 \ , \tag{2.1}$$

where τ and η both take values on the entire real line, i.e. $\tau, \eta \in \mathbb{R}$. Find all three Killing vectors of this spacetime. You'll have to write down the Killing vector equations, and then solve the resulting system of partial differential equations.

Once you have them, show that by taking suitable linear combinations of them, you can end up with three independent Killing vectors L_+, L_0, and L_- that satisfy the commutation relations

$$\begin{aligned} [L_0, L_\pm] &= \pm L_\pm \ , \\ [L_+, L_-] &= L_0 \ . \end{aligned} \tag{2.2}$$

This is the Lie algebra $SL(2, \mathbb{R})$.

4.) **Bonus:** Prove that the Ricci scalar for this spacetime is $R = -2$ at all coordinates (τ, η). This is what we mean when we say that AdS_2 has constant negative curvature. You should find the derivation very similar to our computation of the Ricci scalar on S^2 in problem set 5. The reason for this is that you can actually obtain AdS_2 from the S^2 metric! That is, if we start with the (unit) S^2 metric

$$ds^2_{S^2} = d\theta^2 + \sin^2\theta \, d\phi^2 \ , \tag{2.3}$$

show that if we let $\theta \to i\eta + \frac{\pi}{2}$ and $\phi \to \tau$, then the S^2 metric transforms as

$$ds^2_{S^2} \to \cosh^2\eta \, d\tau^2 - d\eta^2 = -ds^2_{\text{AdS}_2} \ , \tag{2.4}$$

i.e. it becomes the AdS_2 metric, albeit with an overall minus sign.

Important: note that we have not done a true coordinate transformation! The coordinate θ is supposed to be real, as is the coordinate η, and so setting $\theta = i\eta + \frac{\pi}{2}$ clearly violates one of those two assumptions. Additionally, ϕ is only supposed to take values on the interval $[0, 2\pi)$, while τ can take values on the whole real line, which is inconsistent with setting $\phi = \tau$. We have therefore not done a good coordinate transformation; instead, we have done an *analytic continuation*, which is a technique used in complex analysis to extend the domain of a function. In our case, we are extending the domains of the coordinates to the full complex plane and then noticing that there is a particular identification between the S^2 and AdS_2 coordinates that yields the relation (2.4) between the S^2 and AdS_2 metrics.